# Mini-post: The periodic 1D wave equation

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O, many times have I seen the wave equation with periodic boundary conditions:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\varphi(t,x) - \frac{\partial^2}{\partial x^2}\varphi(t,x) = 0,$$

subject to $\varphi(t,0) = \varphi(t,L)$ for all $t$. Don't we all just **know** that the solutions are linear combinations of elements of the form

$$e^{\pm i(k_n x \pm \omega_n t)},$$

with

$$k_n = \frac{2\pi n}{L}, \quad n \in \mathbb{Z},$$

and $\omega_n = |k_n||v|$?

Don't we all **knooooooooow** this?

Well I forgot how to do this and I should be doing other things but here it goes.

## What we all did sometime in the remote past

Separate variables (of course!) by writing $\varphi(t,x) = f(x)g(t)$. Then we have that $\partial_t\varphi(t,x) = f(x)g'(t)$ and $\partial_x\varphi(t,x) = f'(x)g(t)$. Substitute that in:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\varphi(t,x) - \frac{\partial^2}{\partial x^2}\varphi(t,x) = \frac{1}{v^2}f(x)g''(t) - f''(x)g(t).$$

Assume that $f(x)g(t) \neq 0$ and divide the right-hand side by $f(x)g(t)$. Then we have

$$\frac{1}{v^2}\frac{1}{g(t)}g''(t) = \frac{1}{f(x)}f''(x).$$

Now fix some value of $t$, say $t = 0$. This equation implies that for all $x$

$$\frac{1}{f(x)}f''(x) = \frac{1}{v^2}\frac{1}{g(0)}g''(0) = \alpha,$$

where we have **defined** $\alpha$ as the right-hand side. It's clearly a constant. Similarly, if we fix an $x$, say $x = 0$, we have that for **all** $t$, the following equation holds:

$$\frac{1}{v^2}\frac{1}{g(t)}g''(t) = \frac{1}{f(0)}f''(0) = \alpha,$$

where the rightmost equality follows from the previous equation (which holds for all $x$, in particular $x = 0$). Then **both** terms are equal to $\alpha$, a constant to be determined.

Let's try to determine that. Let's work on the equation for $f$. We have that

$$f''(x) = \alpha f(x),$$

which we recognize as a simple second-order homogeneous linear differential equation. The solutions to this equation are of the form

$$f(x) = C_1 e^{\mu x} + C_2 e^{-\mu x},$$

where $C_1, C_2$ are constants to be determined and $\mu^2 = \alpha$. Note that $\mu$ might be complex, depending on whether $\alpha$ is positive or negative (or complex too!). Now the periodic boundary conditions imply $f(0) = f(L)$, so

$$f(0) = C_1 + C_2 = C_1 e^{\mu L} + C_2 e^{-\mu L} = f(L).$$

Save that for later. We also have that $f'(0) = f'(L)$, so

$$f'(0) = \mu C_1 - \mu C_2 = \mu C_1 e^{\mu L} - \mu C_2 e^{-\mu L}.$$

This implies that, assuming that $\mu \neq 0$,

$$C_1 - C_2 = C_1 e^{\mu L} - C_2 e^{-\mu L}.$$

Adding the conditions for $f(0) = f(L)$ and $f'(0) = f'(L)$ we obtain that

$$C_1 = C_1 e^{\mu L},$$

which implies that either $C_1 = 0$ or $e^{\mu L} = 1$. If the first case is true then to avoid the trivial solution, we have to require $C_2 \neq 0$ which implies $e^{-\mu L} = 1$. Either way, this is unavoidable, and it implies that $\mu L = 2\pi n i$ for some $n \in \mathbb{Z}$.

Therefore we can write

$$k_n = \frac{2\pi n}{L}, \quad n \in \mathbb{Z},$$

so that $\mu = i k_n$ and the general solution to $f$ is

$$f(x) = C_1 e^{i k_n x} + C_2 e^{-i k_n x}.$$

Nearly done. Now we work with the equation for $g$:

$$g''(t) = v^2 \alpha g(t).$$

However, $\alpha = \mu^2 = (i k_n)^2 = -k_n^2$, so the general solution is

$$g(t) = K_1 e^{i k_n |v| t} + K_2 e^{-i k_n |v| t}.$$

Here the constants $K_1, K_2$ are left unknown. Now we multiply $g(t)$ by $f(x)$:

$$\varphi(t,x) = f(x)g(t) = C_1 K_1 e^{i(k_n x + k_n |v| t)} + C_1 K_2 e^{i(k_n x - k_n |v| t)} + C_2 K_1 e^{i(-k_n x + k_n |v| t)} + C_2 K_2 e^{i(-k_n x - k_n |v| t)}.$$

Now let $\omega_n = |k_n||v|$. Then the solution is a linear combination of elements of the form

$$e^{\pm i(k_n x \pm \omega_n t)}.$$