Seminar on Differential Geometry
Assignment 7

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Exercise 1

(3 pt) In this exercise we will find a connection on the Hopf bundle. In one of its several incarnations, the Hopf bundle is a principal U(1)-bundle:

$$\mathrm{U}(1) \hookrightarrow S^3 \subset \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}P^1,$$

where the action is given by

$$(z_0, z_1) \cdot \lambda = (\lambda z_0, \lambda z_1),$$

and the projection $\pi : S^3 \to \mathbb{C}P^1$ is given by

$$(z_0, z_1) = (z_0 : z_1).$$

Here, we consider $\mathbb{C}P^1$ as the quotient $\mathbb{C}P^1 = \mathbb{C}^2 - \{0\}/\sim$, where

$$(z_0, z_1) \sim (\lambda z_0, \lambda z_1)$$

for all *non-zero* $\lambda \in \mathbb{C}$. We denote the class of (z_0, z_1) as

$$(z_0 : z_1) = [(z_0, z_1)].$$

Identify $\mathbb{R}^4 \cong \mathbb{C}^2$ via

$$(z_0, z_1) = (x_0, y_0, x_1, y_1),$$

with $z_j = x_j + iy_j$, $j = 0, 1$. Let $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^4$, and define a $\mathfrak{u}(1) = i\mathbb{R}$-valued 1-form ω on S^3 as

$$\omega_p(X) = i \langle X, p \cdot i \rangle,$$

where $X \in T_pS^3$ is interpreted as a vector in $\mathbb{C}^2$. Show that ω is a connection on S^3. Describe the horizontal space H_p (as a subspace of $\mathbb{C}^2$).

Exercise 2

Recall the definition of brackets of valued forms: for $\alpha \in \Omega^i(P, \mathfrak{g})$ and $\beta \in \Omega^j(P, \mathfrak{g})$, define $[\alpha, \beta] \in \Omega^{i+j}(P, \mathfrak{g})$ as

$$[\alpha, \beta] = \sum_{a,b} (\omega^a \wedge \omega^b)[e_a, e_b],$$

where $\{e_1, \ldots, e_m\}$ is a basis for $\mathfrak{g}$ and

$$\alpha = \sum_a \alpha^a e_a \qquad \beta = \sum_b \beta^b e_b.$$

(a) (2 pt) Show that

$$\mathrm{d}[\alpha, \beta] = [\mathrm{d}\alpha, \beta] + (-1)^i [\alpha, \mathrm{d}\beta].$$

(b) (3 pt) Let ω be a connection on a principal G-bundle, with curvature Ω. Prove the **Bianchi identity**:

$$\mathrm{d}\Omega = [\Omega, \omega].$$

Hint: Prove and use the fact that $[[\omega, \omega], \omega] = 0$.

(c) (2 pt) Define the **covariant derivative** $\mathrm{d}^\omega : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g})$ as

$$(\mathrm{d}^\omega \alpha)(X_1, \ldots, X_{k+1}) = \mathrm{d}\alpha\,(X_1^H, \ldots, X_{k+1}^H).$$

Prove that

$$\mathrm{d}^\omega \Omega = 0.$$